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Aharonov–Bohm scattering on parallel flux lines of the same magnitude

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Abstract. The problem of Aharonov-Bohm scattering on parallel flux lines of the same magnitude is solved exactly and the differential cross section is calculated.

1. Introduction

The quantum mechanical scattering of electrons by a flux line was analysed by Aharonov and Bohm (1959). Since then Aharonov-Bohm scattering problems have been solved exactly only for the case of a single flux tube (Aharonov *et al* 1984, Brown 1985, Gauthier and Rochon 1985). In this paper we shall further solve exactly the Aharonov-Bohm scattering on parallel flux lines of the same magnitude. In § 2 we derive a simplified form of the vector potential in elliptical coordinates. In § 3 we solve exactly the Schrödinger equation by means of Mathieu functions. In § 4 we obtain the differential cross section.

2. Vector potential

Let $0XY$ be the coordinate plane perpendicular to two flux lines having coordinates $(a, 0)$ and $(-a, 0)$. We choose two polar coordinates (ρ_1, ϕ_1) and (ρ_2, ϕ_2) with these two points as poles. In the Coulomb gauge, the vector potential is

$$\boldsymbol{A} = \frac{\Phi}{2\pi}\left(\frac{\boldsymbol{e}_{\phi_1}}{\rho_1} + \frac{\boldsymbol{e}_{\phi_2}}{\rho_2}\right) \tag{1}$$

where Φ is the flux of the flux lines and $\boldsymbol{e}_{\phi_1}$ and $\boldsymbol{e}_{\phi_2}$ are the unit vectors in the transverse direction of the two polar coordinates. In terms of rectangular coordinates

$$\boldsymbol{e}_{\phi_1} = \frac{-y\boldsymbol{i} + (x-a)\boldsymbol{j}}{[(x-a)^2 + y^2]^{1/2}} \qquad \boldsymbol{e}_{\phi_2} = \frac{-y\boldsymbol{i} + (x+a)\boldsymbol{j}}{[(x+a)^2 + y^2]^{1/2}}. \tag{2}$$

When we use elliptical coordinates, the transformation equations are

$$x = a\cosh\mu\cos\theta \qquad y = a\sinh\mu\sin\theta \tag{3}$$

the metric coefficients are

$$\begin{aligned} h_\mu &= \left[\left(\frac{\partial x}{\partial\mu}\right)^2 + \left(\frac{\partial y}{\partial\mu}\right)^2\right]^{1/2} = a(\cosh^2\mu - \cos^2\theta)^{1/2} \equiv h \\ h_\theta &= \left[\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2\right]^{1/2} = a(\cosh^2\mu - \cos^2\theta)^{1/2} \equiv h. \end{aligned} \tag{4}$$

The relations between the unit coordinate vectors $\boldsymbol{e}_\mu$, $\boldsymbol{e}_\theta$ and $\boldsymbol{i}, \boldsymbol{j}$ are

$$\begin{aligned} \boldsymbol{i} &= \frac{1}{h}\frac{\partial x}{\partial\mu}\boldsymbol{e}_\mu + \frac{1}{h}\frac{\partial x}{\partial\theta}\boldsymbol{e}_\theta = \frac{a}{h}(\sinh\mu\cos\theta\,\boldsymbol{e}_\mu - \cosh\mu\sin\theta\,\boldsymbol{e}_\theta) \\ \boldsymbol{j} &= \frac{1}{h}\frac{\partial y}{\partial\mu}\boldsymbol{e}_\mu + \frac{1}{h}\frac{\partial y}{\partial\theta}\boldsymbol{e}_\theta = \frac{a}{h}(\cosh\mu\sin\theta\,\boldsymbol{e}_\mu + \sinh\mu\cos\theta\,\boldsymbol{e}_\theta). \end{aligned} \tag{5}$$

In terms of elliptical coordinates (1) becomes

$$\boldsymbol{A} = \frac{\Phi(-\sin\theta\cos\theta\,\boldsymbol{e}_\mu + \sinh\mu\cosh\mu\,\boldsymbol{e}_\theta)}{\pi a(\cosh^2\mu - \cos^2\theta)^{3/2}}. \tag{6}$$

Now we simplify the form of the vector potential by a gauge transformation. The new vector potential is

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda = \frac{1}{h}\left(\frac{-\Phi}{\pi}\frac{\sin\theta\cos\theta}{\cosh^2\mu - \cos^2\theta} + \frac{\partial\Lambda}{\partial\mu}\right)\boldsymbol{e}_\mu + \frac{1}{h}\left(\frac{\Phi}{\pi}\frac{\sinh\mu\cosh\mu}{\cosh^2\mu - \cos^2\theta} + \frac{\partial\Lambda}{\partial\theta}\right)\boldsymbol{e}_\theta. \tag{7}$$

Letting the coefficient of $\boldsymbol{e}_\mu$ be equal to zero, we obtain

$$\frac{\partial\Lambda}{\partial\mu} = \frac{\Phi}{\pi}\frac{\sin\theta\cos\theta}{\cosh^2\mu - \cos^2\theta}. \tag{8}$$

Integrating over μ we obtain

$$\Lambda = \frac{\Phi}{2\pi}\left[\sin^{-1}\left(\frac{\cosh\mu\cos\theta - 1}{\cosh\mu - \cos\theta}\right) + \sin^{-1}\left(\frac{\cosh\mu\cos\theta + 1}{\cosh\mu + \cos\theta}\right) + 2g(\theta)\right] \tag{9}$$

where $g(\theta)$ is an arbitrary function of θ. Substituting (9) into (7) we obtain

$$\boldsymbol{A}' = \frac{\phi}{\pi}\frac{g'(\theta)}{h}\boldsymbol{e}_\theta = \frac{\Phi g'(\theta)}{\pi a(\cosh^2\mu - \cos^2\theta)^{1/2}}\boldsymbol{e}_\theta \qquad g'(\theta) \equiv \mathrm{d}g(\theta)/\mathrm{d}\theta. \tag{10}$$

Equation (10) must satisfy the physical requirement that

$$\oint_{C_1}\boldsymbol{A}'\cdot\mathrm{d}\boldsymbol{r} = \Phi \qquad \oint_{C_2}\boldsymbol{A}'\cdot\mathrm{d}\boldsymbol{r} = \Phi \tag{11}$$

where C_1 and C_2 are two closed paths around each flux. If we choose C_1 and C_2 as shown in figure 1 then (11) becomes

$$\frac{\Phi}{\pi}\int_{-\pi/2}^{\pi/2} g'(\theta)\,\mathrm{d}\theta = \Phi \qquad \frac{\phi}{\pi}\int_{\pi/2}^{3\pi/2} g'(\theta)\,\mathrm{d}\theta = \Phi \tag{12}$$

since

$$\mathrm{d}\boldsymbol{r} = h\,\mathrm{d}\mu\,\boldsymbol{e}_\mu + h\,\mathrm{d}\theta\,\boldsymbol{e}_\theta \tag{13}$$

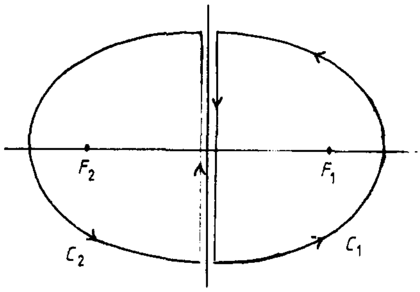

Figure 1. Two closed paths around each flux.

and $d\theta = 0$ along the y axis. The simplest choice of $g'(\theta)$ is

$$g'(\theta) = 1. \tag{14}$$

Substituting (14) into (10) we obtain

$$\boldsymbol{A}' = \frac{\Phi}{\pi a(\cosh^2 \mu - \cos^2 \theta)^{1/2}} \boldsymbol{e}_\theta. \tag{15}$$

3. The Schrödinger equation

The Schrödinger equation is

$$\left(\nabla - \frac{\mathrm{i}e}{\hbar c} \boldsymbol{A}'\right)^2 \psi' = -k^2 \psi'. \tag{16}$$

Substituting (15) into (16) we obtain

$$\frac{\partial^2 \psi'}{\partial \mu^2} + \frac{\partial^2 \psi'}{\partial \theta^2} + \mathrm{i}4\alpha \frac{\partial \psi'}{\partial \theta} - [4\alpha^2 - k^2 a^2(\cosh^2 \mu - \cos^2 \theta)]\psi' = 0 \tag{17}$$

where $k \equiv (2mE/\hbar^2)^{1/2}$ is the wavenumber and $\alpha \equiv -e\Phi/2\pi\hbar c$ is the quantum number of the flux. By writing $\psi' = M(\mu)\Theta(\theta)$ we get

$$\frac{M'' + a^2k^2(\cosh^2 \mu)M}{M} = -\frac{\Theta'' + \mathrm{i}4\alpha\Theta' - (4\alpha^2 + a^2k^2\cos^2\theta)\Theta}{\Theta} = \lambda + 2q \tag{18}$$

where $q \equiv a^2k^2/4$ and $\lambda + 2q$ is the constant introduced in separating variables. Let

$$\nu = \mathrm{i}\mu \qquad \Theta(\theta) = \mathrm{e}^{-\mathrm{i}2\alpha\theta} Q(\theta) \tag{19}$$

then (18) becomes

$$\begin{aligned} &\mathrm{d}^2M/\mathrm{d}\nu^2 + (\lambda - 2q\cos 2\nu)M = 0 \\ &\mathrm{d}^2Q/\mathrm{d}\theta^2 + (\lambda - 2q\cos 2\theta)Q = 0 \end{aligned} \tag{20}$$

which are recognised as the Mathieu equations. The wavefunction ψ corresponding to $\boldsymbol{A}$ is related by the wavefunction ψ' corresponding to $\boldsymbol{A}'$ by

$$\psi = \psi' \exp\left(-\mathrm{i}\frac{e}{\hbar c}\Lambda\right)$$

$$= \psi' \exp\left\{\mathrm{i}\alpha\left[\sin^{-1}\left(\frac{\cosh\mu\cos\theta - 1}{\cosh\mu - \cos\theta}\right)\right.\right.$$

$$\left.\left. + \sin^{-1}\left(\frac{\cosh\mu\cos\theta + 1}{\cosh\mu + \cos\theta}\right) + 2\theta - \pi\right]\right\} \tag{21}$$

where $g(\theta)$ in (9) is chosen to be $(\theta - \pi/2)$ and hence (14) is satisfied. Using the general solution of (20) (Mclachlan 1947) we can obtain the general solution of ψ':

$$\psi = \exp\left(-\mathrm{i}\frac{e}{\hbar c}\Lambda - \mathrm{i}2\alpha\theta\right)\sum_{n=0}^{\infty}\{[A_n Ce_{2n}(\mu, q) + \bar{A}_n Fey_{2n}(\mu, q)]ce_{2n}(\theta, q)$$

$$+ [B_n Ce_{2n+1}(\mu, q) + \bar{B}_n Fey_{2n+1}(\mu, q)]ce_{2n+1}(\theta, q)$$

$$+ [C_n Se_{2n+1}(\mu, q) + \bar{C}_n Gey_{2n+1}(\mu, q)]se_{2n+1}(\theta, q)$$

$$+ [D_n Se_{2n+2}(\mu, q) + \bar{D}_n Gey_{2n+2}(\mu, q)]se_{2n+2}(\theta, q)\}. \tag{22}$$

Equation (22) can be rewritten as

$$\psi = \sum_{m=0}^{\infty}\sum_{l}[C^c_{ml}Ce_l(\mu, q) + \bar{C}^c_{ml}Fey_l(\mu, q) + S^c_{ml}Se_l(\mu, q) + \bar{S}^c_{ml}Gey_l(\mu, q)]ce_m(\theta, q)$$

$$+ \sum_{m=1}^{\infty}\sum_{l}[C^s_{ml}Ce_l(\mu, q) + \bar{C}^s_{ml}Fey_l(\mu, q) + S^s_{ml}Se_l(\mu, q)$$

$$+ \bar{S}^s_{ml}Gey_l(\mu, q)]se_m(\theta, q). \tag{23}$$

It should be noted that coefficients $C^c_{ml}, \bar{C}^c_{ml}, S^c_{ml}, \ldots,$ are functions of α.

Now we shall find these coefficients under the conditions $\mu \to \infty$ and $q \to 0$. When $\mu \to \infty$, we have

$$\Lambda = \frac{\Phi}{2\pi}[\sin^{-1}(\cos\theta) + \sin^{-1}(\cos\theta) + 2\theta - \pi] = 0$$
$$\psi = \psi' \exp\left(-\mathrm{i}\frac{e}{\hbar c}\Lambda\right) = \psi' \tag{24}$$

$$\cosh\mu \to \tfrac{1}{2}\mathrm{e}^{\mu} \qquad \tfrac{1}{2}a\,\mathrm{e}^{\mu} \to \rho \qquad \theta \to \phi \qquad \text{hence } \boldsymbol{A}' \to \Phi\boldsymbol{e}_{\phi}/\pi\rho \tag{25}$$

$$\begin{aligned} &Ce_l(\mu, q) \to p'_l J_l(k\rho) && l \geq 0\\ &Se_l(\mu, q) \to s'_l J_l(k\rho) && l \geq 1\\ &Fey_l(\mu, q) \to p'_l Y_l(k\rho) && l \geq 0\\ &Gey_l(\mu, q) \to s'_l Y_l(k\rho) && l \geq 1 \end{aligned} \tag{26}$$

where (ρ, ϕ) are polar coordinates with the origin 0 of the rectangular coordinates as pole, the constant multipliers p'_l and s'_l are given by Mclachlan (1947, pp 368–9). When $q \to 0$,

$$ce_m(\theta, q) \to \cos(m\theta) \qquad se_m(\theta, q) \to \sin(m\theta). \tag{27}$$

When $\mu \to \infty$ then $a \cosh \mu \sim a \sinh \mu \sim \frac{1}{2} a\, e^{\mu} \sim \rho$ and we have $M(\mu) = R(\rho)$. Letting $\Theta = e^{im\phi}$ we obtain

$$R'' + \frac{1}{\rho} R' + \left(k^2 - \frac{(m+2\alpha)^2}{\rho^2} \right) R = 0. \tag{28}$$

The solutions of (28) are Bessel functions of fractional order. Let τ be the angle between the y axis and the wavevector $\boldsymbol{k}$ of the incident wave, then we have (see Aharonov and Bohm (1959) who chose $\tau = -\pi/2$):

$$\psi = \sum_{m=0}^{\infty} e^{-i\alpha\pi + im\tau} J_{m+2\alpha}\, e^{im\phi} + \sum_{m=1}^{\infty} (-1)\, e^{i\alpha\pi - im\tau} J_{m-2\alpha}\, e^{-im\phi}. \tag{29}$$

By means of the asymptotic relations of Bessel functions we can write

$$J_{m\pm 2\alpha}(k\rho) = \tfrac{1}{2} e^{\mp i\alpha\pi} [J_m(k\rho) + i\, Y_m(k\rho)] + \tfrac{1}{2} e^{\pm i\alpha\pi} [J_m(k\rho) - i\, Y_m(k\rho)]. \tag{30}$$

Substituting (30) into (29) we obtain

$$\begin{aligned}
\psi = {} & (e^{-i2\alpha\pi} + 1) J_0(k\rho) + i(e^{-i2\alpha\pi} - 1) Y_0(k\rho) \\
& + \sum_{n=1}^{\infty} \{4 \cos(2n\tau - \alpha\pi) \cos(\alpha\pi) J_{2n}(k\rho) \\
& + i4 \sin(2n\tau - \alpha\pi) \sin(\alpha\pi) Y_{2n}(k\rho)\} \cos(2n\phi) \\
& + \sum_{n=0}^{\infty} \{i4 \sin[(2n+1)\tau - \alpha\pi] \cos(\alpha\pi) J_{2n+1}(k\rho) \\
& + 4 \cos[(2n+1)\tau - \alpha\pi] \sin(\alpha\pi) Y_{2n+1}(k\rho)\} \cos[(2n+1)\phi] \\
& + \sum_{n=0}^{\infty} \{i4 \cos[(2n+1)\tau - \alpha\pi] \cos(\alpha\pi) J_{2n+1}(k\rho) \\
& - 4 \sin[(2n+1)\tau - \alpha\pi] \sin(\alpha\pi) Y_{2n+1}(k\rho)\} \sin[(2n+1)\phi] \\
& + \sum_{n=0}^{\infty} \{-4 \sin[(2n+2)\tau - \alpha\pi] \cos(\alpha\pi) J_{2n+2}(k\rho) \\
& + i4 \cos[(2n+2)\tau - \alpha\pi] \sin(\alpha\pi) Y_{2n+2}(k\rho)\} \sin[(2n+2)\phi].
\end{aligned} \tag{31}$$

In the limit $\mu \to \infty$ and $q \to 0$, by using (26), (27) and $\theta \to \phi$, we obtain from (23) the following formula for ψ:

$$\begin{aligned}
\psi = {} & [(e^{-i2\alpha\pi} + 1) Ce_0(\mu, q)/2p'_0 + i(e^{-i2\alpha\pi} - 1) Fey_0(\mu, q)/2p'_0] ce_0(\theta, q) \\
& + \sum_{n=1}^{\infty} \{2 \cos(2n\tau - \alpha\pi) \cos(\alpha\pi) Ce_{2n}(\mu, q)/p'_{2n} \\
& + 2i \sin(2n\tau - \alpha\pi) \sin(\alpha\pi) Fey_{2n}(\mu, q)/p'_{2n}\} ce_{2n}(\theta, q) \\
& + \sum_{n=0}^{\infty} \{2i \sin[(2n+1)\tau - \alpha\pi] \cos(\alpha\pi) Ce_{2n+1}(\mu, q)/p'_{2n+1} \\
& + 2 \cos[(2n+1)\tau - \alpha\pi] \sin(\alpha\pi) Fey_{2n+1}(\mu, q)/p'_{2n+1}\} ce_{2n+1}(\theta, q) \\
& + \sum_{n=0}^{\infty} \{2i \cos[(2n+1)\tau - \alpha\pi] \cos(\alpha\pi) Se_{2n+1}(\mu, q)/s'_{2n+1} \\
& - 2 \sin[(2n+1)\tau - \alpha\pi] \sin(\alpha\pi) Gey_{2n+1}(\mu, q)/s'_{2n+1}\} se_{2n+1}(\theta, q) \\
& + \sum_{n=0}^{\infty} \{-2 \sin[(2n+2)\tau - \alpha\pi] \cos(\alpha\pi) Se_{2n+2}(\mu, q)/s'_{2n+2} \\
& + 2i \cos[(2n+2)\tau - \alpha\pi] \sin(\alpha\pi) Gey_{2n+2}(\mu, q)/s'_{2n+2}\} se_{2n+2}(\theta, q).
\end{aligned} \tag{32}$$

4. Scattering cross section

Since in the asymptotic region $\phi=\theta$, (29) can be rewritten as

$$\psi=\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]+f(\theta)\,\mathrm{e}^{\mathrm{i}k\rho}/\sqrt{k\rho}. \tag{33}$$

By the orthogonality of Mathieu functions we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]y_j(\theta,q)\,\mathrm{d}\theta+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)y_j(\theta,q)\,\mathrm{d}\theta$$
$$=\frac{1}{\pi}\int_{-\pi}^{\pi}\psi y_j(\theta,q)\,\mathrm{d}\theta \qquad j=0,1,2,3,4 \tag{34}$$

where $y_0(\theta,q)=ce_0(\theta,q)$, $y_1(\theta,q)=ce_{2n}(\theta,q)$, $y_2(\theta,q)=ce_{2n+1}(\theta,q)$, $y_3(\theta,q)=se_{2n+1}(\theta,q)$, $y_4(\theta,q)=se_{2n+2}(\theta,q)$. The terms in (34) can be rewritten as

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\exp[-2\mathrm{i}\alpha\theta+\mathrm{i}k\rho\sin(\theta+\tau)]y_j(\theta,q)\,\mathrm{d}\theta$$
$$=G_j=G_j^+\mathrm{e}^{\mathrm{i}k\rho}/\sqrt{k\rho}+G_j^-\mathrm{e}^{-\mathrm{i}k\rho}/\sqrt{k\rho} \tag{35}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)y_j(\theta,q)\,\mathrm{d}\theta=F_j \tag{36}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\psi y_j(\theta,q)\,\mathrm{d}\theta=H_j=H_j^+\mathrm{e}^{\mathrm{i}k\rho}/\sqrt{k\rho}+H_j^-\mathrm{e}^{-\mathrm{i}k\rho}/\sqrt{k\rho}. \tag{37}$$

Substituting (35)-(37) into (34), then comparing the coefficients of $\mathrm{e}^{\mathrm{i}k\rho}\sqrt{k\rho}$, we can find $F_j(\alpha,\tau)$:

$$F_j=H_j^+-G_j^+. \tag{38}$$

Since $y_j(\theta,q)$ form a complete set, we can express $f(\theta)$ as

$$f(\theta)=\tfrac{1}{2}F_0ce_0(\theta,q)+\sum_{n=1}^{\infty}F_1ce_{2n}(\theta,q)+\sum_{n=0}^{\infty}F_2ce_{2n+1}(\theta,q)$$
$$+\sum_{n=0}^{\infty}F_3se_{2n+1}(\theta,q)+\sum_{n=0}^{\infty}F_4se_{2n+2}(\theta,q) \tag{39}$$

where we have used the normalisation conditions of Mathieu functions. It should be pointed out, when we substitute (35)-(37) into (34), that the coefficient of $\mathrm{e}^{-\mathrm{i}k\rho}/\sqrt{k\rho}$ is

$$\tfrac{1}{2}(H_0^--G_0^-)+\sum_{n=1}^{\infty}(H_1^--G_1^-)ce_{2n}(\theta,q)+\sum_{j=2}^{4}\sum_{n=0}^{\infty}(H_j^--G_j^-)y_j \tag{40}$$

which can be proved to be equal to zero (see appendix 1). Now let us calculate the terms G_j in (35) and the terms H_j in (37).

4.1. Calculation of G_j

Using formulae

$$\mathrm{e}^{\mathrm{i}k\rho\sin(\theta+\tau)}=\sum_{m=-\infty}^{\infty}J_m(k\rho)\,\mathrm{e}^{\mathrm{i}m(\theta+\tau)} \tag{41}$$

and

$$y_1 = ce_{2n}(\theta, q) = \sum_{r=0}^{\infty} A_{2r}^{2n} \cos(2r\theta) \tag{42}$$

we get

$$G_1 = \sum_{m=-\infty}^{\infty} e^{im\tau} J_m(k\rho) \sum_{r=0}^{\infty} A_{2r}^{2n} \frac{1}{\pi} \int_{-\pi}^{\pi} e^{i(m-2\alpha)\theta} \cos(2r\theta)\, d\theta. \tag{43}$$

Using the asymptotic approximation

$$J_m(k\rho) \sim (e^{i(k\rho - m\pi/2 - \pi/4)} + e^{-i(k\rho - m\pi/2-\pi/4)})/\sqrt{2\pi k\rho} \tag{44}$$

and the formula

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{i(m-2\alpha)\theta} \cos(2r\theta)\, d\theta = \frac{\sin(2\alpha\pi)}{\pi} (-1)^{m+1} \left(\frac{1}{m-2\alpha+2r} + \frac{1}{m-2\alpha-2r}\right) \tag{45}$$

(43) can be written as

$$G_1 = \frac{\sin(2\alpha\pi)}{\pi\sqrt{2\pi k\rho}} \left(e^{i(k\rho-\pi/4)} \sum_{r=0}^{\infty} A_{2r}^{2n} g_-(\tau, r) + e^{-i(k\rho-\pi/4)} \sum_{r=0}^{\infty} A_{2r}^{2n} g_+(\tau, r) \right) \tag{46}$$

where

$$\begin{aligned} g_\pm(\tau, r) &\equiv \sum_{m=-\infty}^{\infty} e^{im(\tau\pm\pi/2)} (-1)^{m+1} \left(\frac{1}{m-2\alpha+2r} + \frac{1}{m-2\alpha-2r}\right) \\ &= \frac{\pi\, e^{i2\alpha(\tau\pm\pi/2)}}{\sin(2\alpha\pi)} (-1)^l 2\cos[l(\tau\pm\pi/2)]. \end{aligned} \tag{47}$$

Substituting (47) into (46) we get

$$G_1 = \left(\frac{2}{\pi}\right)^{1/2} ce_{2n}(\tau+\pi/2, q)\, e^{i2\alpha\tau} \left(e^{-i\alpha\pi - i\pi/4} \frac{e^{ik\rho}}{\sqrt{k\rho}} + e^{i\alpha\pi + i\pi/4} \frac{e^{-ik\rho}}{\sqrt{k\rho}} \right). \tag{48}$$

Putting $n = 0$ in (48) we obtain G_0. Similarly we can obtain

$$G_2 = \left(\frac{2}{\pi}\right)^{1/2} ce_{2n+1}(\tau+\pi/2, q)\, e^{i2\alpha\tau} \left(-e^{-i\alpha\pi - i\pi/4} \frac{e^{ik\rho}}{\sqrt{k\rho}} + e^{i\alpha\pi + i\pi/4} \frac{e^{-ik\rho}}{\sqrt{k\rho}} \right) \tag{49}$$

$$G_3 = \left(\frac{2}{\pi}\right)^{1/2} se_{2n+1}(\tau+\pi/2, q)\, e^{i2\alpha\tau} \left(-e^{-i\alpha\pi - i\pi/4} \frac{e^{ik\rho}}{\sqrt{k\rho}} + e^{i\alpha\pi + i\pi/4} \frac{e^{-ik\rho}}{\sqrt{k\rho}} \right) \tag{50}$$

$$G_4 = \left(\frac{2}{\pi}\right)^{1/2} se_{2n+2}(\tau+\pi/2, q)\, e^{i2\alpha\tau} \left(e^{-i\alpha\pi - i\pi/4} \frac{e^{ik\rho}}{\sqrt{k\rho}} + e^{i\alpha\pi + i\pi/4} \frac{e^{-ik\rho}}{\sqrt{k\rho}} \right). \tag{51}$$

4.2. Calculation of H_j

Using (26), (32), (44) and the asymptotic approximation of $Y_m(k\rho)$:

$$Y_m(k\rho) \sim (e^{i(k\rho - m\pi/2 - \pi/4)} - e^{-i(k\rho - m\pi/2-\pi/4)})/\sqrt{2\pi k\rho} \tag{52}$$

we obtain

$$\begin{aligned} H_0 &= \frac{1}{\sqrt{2\pi}} \left(2\, e^{-i2\alpha\pi - i\pi/4} \frac{e^{ik\rho}}{\sqrt{k\rho}} + e^{i\pi/4} \frac{e^{-ik\rho}}{\sqrt{k\rho}} \right) \\ &= H_0^+ \frac{e^{ik\rho}}{\sqrt{k\rho}} + H_0^- \frac{e^{-ik\rho}}{\sqrt{k\rho}} \end{aligned} \tag{53}$$

$$H_1 = \frac{(-1)^n}{\sqrt{2\pi}}\left(2\cos(2n\tau - 2\alpha\pi)\,\mathrm{e}^{-\mathrm{i}\pi/4}\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + \mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi/4}2\cos(2n\tau)\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}\right)$$

$$= H_1^+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_1^-\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{54}$$

$$H_2 = \frac{(-1)^n}{\sqrt{2\pi}}\left(2\mathrm{i}\sin[(2n+1)\tau - 2\alpha\pi]\,\mathrm{e}^{-\mathrm{i}\pi/4}\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} - 2\,\mathrm{e}^{\mathrm{i}\pi/4}\sin[(2n+1)\tau]\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}\right)$$

$$= H_2^+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_2^-\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{55}$$

$$H_3 = \frac{(-1)^n}{\sqrt{2\pi}}\left(2\mathrm{i}\cos[(2n+1)\tau - 2\alpha\pi]\,\mathrm{e}^{-\mathrm{i}\pi/4}\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} - 2\,\mathrm{e}^{\mathrm{i}\pi/4}\cos[(2n+1)\tau]\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}\right)$$

$$= H_3^+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_3^-\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}} \tag{56}$$

$$H_4 = \frac{(-1)^{n+1}}{\sqrt{2\pi}}\left(-2\sin[(2n+2)\tau - 2\alpha\pi]\,\mathrm{e}^{-\mathrm{i}\pi/4}\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} - 2\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi/4}\sin[(2n+2)\tau]\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}\right)$$

$$= H_4^+\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}} + H_4^-\frac{\mathrm{e}^{-\mathrm{i}k\rho}}{\sqrt{k\rho}}. \tag{57}$$

Using the above results we can obtain F_j from (38), and hence obtain $f(\theta)$ from (39). In appendix 1 we prove that the summation of all the terms involving G_0^+, G_1^+, G_2^+, G_3^+, G_4^+ equals zero and hence we obtain

$$f(\theta) = \tfrac{1}{2}H_0^+ ce_0(\theta, q) + \sum_{n=1}^{\infty} H_1^+ ce_{2n}(\theta, q) + \sum_{n=0}^{\infty} H_2^+ ce_{2n+1}(\theta, q)$$

$$+ \sum_{n=0}^{\infty} H_3^+ se_{2n+1}(\theta, q) + \sum_{n=0}^{\infty} H_4^+ se_{2n+2}(\theta, q). \tag{58}$$

4.3. *The case when q is small*

In this case we can expand $y_j(\theta, q)$ as a power series of q, and so we can do the same thing for $f(\theta)$. From (58) we find the term not containing q is

$$f_0(\theta) = \frac{\mathrm{e}^{-\mathrm{i}3\pi/4}}{\sqrt{2\pi}}\sin(2\alpha\pi)\exp\left[-\mathrm{i}\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right)\right]\left[\cos\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right)\right]^{-1} \tag{59}$$

and the term containing the first power of q is

$$f_1(\theta) = \frac{q\,\mathrm{e}^{-\mathrm{i}\pi/4}}{2\sqrt{2\pi}}\{\cos(2\theta - 2\alpha\pi) - \mathrm{e}^{-\mathrm{i}2\alpha\pi}\cos\theta$$

$$-\cos(\tau-\theta)[(\pi/2+\tau+\theta)\cos(2\alpha\pi) - \sin(2\alpha\pi)$$

$$\times(\cosh^{-1}|\sec(\tau+\theta)| + \ln|2\cos(\tau+\theta)|)]\}. \tag{60}$$

The detailed derivation of (59) and (60) is given in appendix 2. In short

$$f(\theta) = f_0(\theta) + f_1(\theta) + \mathrm{O}(q^2). \tag{61}$$

When $q = 0$ and $\tau = -\pi/2$,

$$f_0(\theta) = \frac{\mathrm{e}^{-\mathrm{i}3\pi/4}}{\sqrt{2\pi}}\sin(2\alpha\pi)\frac{\mathrm{e}^{-\mathrm{i}\theta/2}}{\cos(\theta/2)} \tag{62}$$

we obtain the result of Aharonov and Bohm (1959) as expected, the only difference being the replacement of α by 2α.

Neglecting $O(q^2)$ we obtain the scattering cross section

$$\begin{aligned}\sigma &= |f(\theta)|^2 = (\operatorname{Re} f(\theta))^2 + (\operatorname{Im} f(\theta))^2 \\ &= \frac{\sin^2(2\alpha\pi)}{2\pi}\cos^{-2}\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right) \\ &\quad - q\frac{\sin(2\alpha\pi)}{2\pi}\bigg\{\cos\theta\sin(2\alpha\pi) \\ &\quad + \tan\left(\frac{\theta+\tau}{2}+\frac{\pi}{4}\right)[\cos(2\theta-2\alpha\pi)-\cos(2\alpha\pi)\cos\theta] \\ &\quad - \cos(\tau-\theta)\left[\left(\frac{\pi}{2}+\tau+\theta\right)\cos(2\alpha\pi)\right. \\ &\quad \left.\left. - \sin(2\alpha\pi)(\cosh^{-1}|\sec(\tau+\theta)| + \ln|2\cos(\tau+\theta)|)\right]\right\}. \end{aligned} \tag{63}$$

When $\tau = -\pi/2$ and $2\alpha = n + \frac{1}{2}$, (63) reduces to

$$\begin{aligned}\sigma = |f(\theta)|^2 &= \frac{1}{2\pi\cos^2(\theta/2)} - \frac{q}{2\pi}[2\cos\theta - \cos^2\theta - 2\sin^2(\theta/2) \\ &\quad \times(\cosh^{-1}|\operatorname{cosec}\theta| + \ln|2\sin\theta|)]. \end{aligned} \tag{64}$$

In this case, the dependence of σ on θ for $q = 0$ and $q = 0.1$ are shown in figure 2.

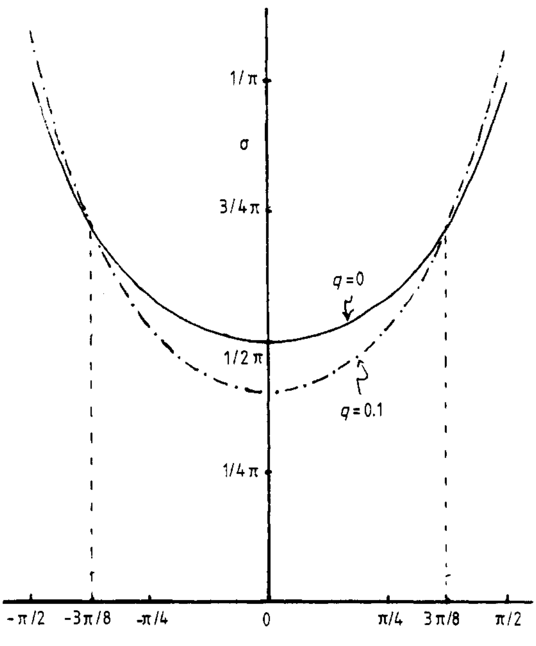

Figure 2. Dependence of σ on θ.

4.4. The case when $q \gg 1$

When $q \to \infty$ we know (Mclachlan 1947) that

$$A_{2r}^{2n}/A_0^{2n} \to (-1)^r 2 \qquad A_{2r+1}^{2n+1}/A_1^{2n+1} \to (-1)^r(2r+1)$$
$$B_{2r+1}^{2n+1}/B_1^{2n+1} \to (-1)^r \qquad B_{2r+2}^{2n+2}/B_2^{2n+2} \to (-1)^r(r+1) \tag{65}$$
$$A_0^{2n} \to 0 \qquad A_1^{2n+1} \to 0 \qquad B_1^{2n+1} \to 0 \qquad B_2^{2n+2} \to 0.$$

Hence in this case from (58) we obtain the result

$$f(\theta) \to 0 \tag{66}$$

which is obvious from the physical point of view.

When q is large but not yet infinite, we can use the asymptotic formulae for ce_m and se_m when $q > 0$ is large enough. For simplicity, we only write out the result when $\tau = -\pi/2$ and $\theta \approx 0$:

$$\sigma = |f|^2 = \frac{2\cos^2(2\alpha\pi)}{\pi^2 q^{1/2}}\left[\left(\tfrac{1}{2}p_0' + \sum_{n=1}^{\infty} p_{2n}'\right)^2 + \left(\tfrac{1}{2}\tan(2\alpha\pi)p_0' + \sum_{n=0}^{\infty} p_{2n+1}'\right)^2\right] \tag{67}$$

where

$$\begin{aligned} p_{2n}' &= (-1)^n ce_{2n}(0,q)ce_{2n}(\pi/2,q)/A_0^{(2n)} \\ p_{2n+1}' &= (-1)^{n+1} ce_{2n+1}(0,q)ce_{2n+1}'(\pi/2,q)/q^{1/2}A_1^{(2n+1)}. \end{aligned} \tag{68}$$

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Appendix 1

Here we give the proof of the following equations:

$$\tfrac{1}{2}G_0^+ ce_0(\theta,q) + \sum_{n=1}^{\infty} G_1^+ ce_{2n}(\theta,q) + \sum_{n=0}^{\infty} G_2^+ ce_{2n+1}(\theta,q) + \sum_{n=0}^{\infty} G_3^+ se_{2n+1}(\theta,q)$$
$$+ \sum_{n=0}^{\infty} G_4^+ se_{2n+2}(\theta,q) = 0 \tag{A1.1}$$

$$\tfrac{1}{2}G_0^- ce_0(\theta,q) + \sum_{n=1}^{\infty} G_1^- ce_{2n}(\theta,q) + \sum_{n=0}^{\infty} G_2^- ce_{2n+1}(\theta,q) + \sum_{n=0}^{\infty} G_3^- se_{2n+1}(\theta,q)$$
$$+ \sum_{n=0}^{\infty} G_4^- se_{2n+2}(\theta,q) = 0 \tag{A1.2}$$

$$\tfrac{1}{2}H_0^- ce_0(\theta,q) + \sum_{n=1}^{\infty} H_1^- ce_{2n}(\theta,q) + \sum_{n=0}^{\infty} H_2^- ce_{2n+1}(\theta,q) + \sum_{n=0}^{\infty} H_3^- se_{2n+1}(\theta,q)$$
$$+ \sum_{n=0}^{\infty} H_4^- se_{2n+2}(\theta,q) = 0. \tag{A1.3}$$

Let us consider the case when q is small, then $ce_0(z, q) = 1$. Firstly let us calculate

$$\Sigma_{0,1} \equiv \tfrac{1}{2} + \sum_{n=1}^{\infty} ce_{2n}(\tau + \pi/2, q) ce_{2n}(\theta, q). \tag{A1.4}$$

Using the formulae

$$ce_{2n}(\theta, q) = \sum_{r=0}^{\infty} A_{2r}^{2n} \cos(2r\theta) \qquad \cos(2r\theta) = \sum_{n=0}^{\infty} A_{2r}^{2n} ce_{2n}(\theta, q) \tag{A1.5}$$

we obtain

$$\begin{aligned}\Sigma_{0,1} &= \sum_{n=0}^{\infty} ce_{2n}(\tau + \pi/2, q) \sum_{r=0}^{\infty} A_{2r}^{2n} \cos(2r\theta) - \tfrac{1}{2} \\ &= \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} A_{2r}^{2n} ce_{2n}\, (\tau + \pi/2, q) \cos(2r\theta) - \tfrac{1}{2} \\ &= \sum_{r=0}^{\infty} \cos[2r(\tau + \pi/2)] \cos(2r\theta) - \tfrac{1}{2} = 0. \end{aligned} \tag{A1.6}$$

Similarly we get

$$\begin{aligned}\Sigma_2 &\equiv \sum_{n=0}^{\infty} ce_{2n+1}(\tau + \pi/2, q) ce_{2n+1}(\theta, q) = 0 \\ \Sigma_3 &\equiv \sum_{n=0}^{\infty} se_{2n+1}(\tau + \pi/2, q) se_{2n+1}(\theta, q) = 0 \\ \Sigma_4 &\equiv \sum_{n=0}^{\infty} se_{2n+2}(\tau + \pi/2, q) se_{2n+2}(\theta, q) = 0. \end{aligned} \tag{A1.7}$$

Since the LHS of (A1.1) is equal to

$$\left(\frac{2}{\pi}\right)^{1/2} \mathrm{e}^{\mathrm{i}2\alpha\tau}\, \mathrm{e}^{-\mathrm{i}\alpha\pi - \mathrm{i}\pi/4} (\Sigma_{0,1} - \Sigma_2 - \Sigma_3 + \Sigma_4) \tag{A1.8}$$

then we prove (A1.1) by (A1.6) and (A1.7). Similarly, since the LHS of (A1.2) is equal to

$$\left(\frac{2}{\pi}\right)^{1/2} \mathrm{e}^{\mathrm{i}2\alpha\tau}\, \mathrm{e}^{\mathrm{i}\alpha\pi + \mathrm{i}\pi/4} (\Sigma_{0,1} + \Sigma_2 + \Sigma_3 + \Sigma_4) \tag{A1.9}$$

we prove (A1.2). By aid of (A1.5) and similar equations, the LHS of (A1.3) can be written as

$$\frac{\mathrm{e}^{\mathrm{i}\pi/4}}{\sqrt{2\pi}} (\Sigma_{0,1} + \Sigma_2 + \Sigma_3 + \Sigma_4) \tag{A1.10}$$

and we thus prove (A1.3). Using (A1.2) and (A1.3) we prove that the coefficient (40) is equal to zero. With the aid of (A1.1) we obtain (58).

Appendix 2. Derivation of (59) and (60)

In (58) we use the following expansion formulae:

$$ce_{2n}(\theta, q) = \cos(2n\theta) - q\left(\frac{\cos[2(n+1)\theta]}{4(2n+1)} - \frac{\cos[2(n-1)\theta]}{4(2n-1)}\right) + \mathrm{O}(q^2) \tag{A2.1}$$

$$ce_{2n+1}(\theta, q) = \cos[(2n+1)\theta] - q\left(\frac{\cos[(2n+3)\theta]}{4(2n+2)} - \frac{\cos[(2n-1)\theta]}{4\times 2n}\right) + O(q^2) \tag{A2.2}$$

$$se_{2n+1}(\theta, q) = \sin[(2n+1)\theta] - q\left(\frac{\sin[(2n+3)\theta]}{4(2n+2)} - \frac{\sin[(2n-1)\theta]}{4\times 2n}\right) + O(q^2) \tag{A2.3}$$

$$se_{2n+2}(\theta, q) = \sin[(2n+2)\theta] - q\left(\frac{\sin[(2n+4)\theta]}{4(2n+3)} - \frac{\sin(2n\theta)}{4(2n+1)}\right) + O(q^2). \tag{A2.4}$$

Then, collecting the terms not containing q, through quite a tedious calculation, we obtain (59):

$$\begin{aligned} f_0(\theta) &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\left(e^{-i2\alpha\pi} - \frac{\cos(\theta+\tau-2\alpha\pi)}{\cos(\theta+\tau)} - \frac{\sin(2\alpha\pi)}{\cos(\theta+\tau)}\right) \\ &= \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\sin(2\alpha\pi)\left(\frac{-i\cos(\theta+\tau)-[1+\sin(\theta+\tau)]}{\cos(\theta+\tau)}\right) \\ &= \frac{-i\,e^{-i\pi/4}}{\sqrt{2\pi}}\sin(2\alpha\pi)\left(\frac{\sin(\theta+\tau+\pi/2)-i[1-\cos(\theta+\tau+\pi/2)]}{\sin(\theta+\tau+\pi/2)}\right) \\ &= \frac{e^{-i3\pi/4}}{\sqrt{2\pi}}\sin(2\alpha\pi)\frac{\exp[-i(\theta/2+\tau/2+\pi/4)]}{\cos(\theta/2+\tau/2+\pi/4)}. \end{aligned} \tag{A2.5}$$

When we collect the terms containing the first power of q, the result is

$$\begin{aligned} f_1(\theta) = \frac{q\,e^{-i\pi/4}}{\sqrt{2\pi}}\Bigg[& -\tfrac{1}{2}e^{-i2\alpha\pi}\cos\theta \\ & + \sum_{n=1}^{\infty}(-1)^{n+1}2\cos(2n\tau-2\alpha\pi)\left(\frac{\cos[2(n+1)\theta]}{4(2n+1)} - \frac{\cos[2(n-1)\theta]}{4(2n-1)}\right) \\ & -\tfrac{1}{4}\sin(\tau-2\alpha\pi)\cos(3\theta) + \sum_{n=1}^{\infty}(-1)^{n+1}2\sin[(2n+1)\tau-2\alpha\pi] \\ & \times\left(\frac{\cos[(2n+3)\theta]}{4(2n+2)} - \frac{\cos[(2n-1)\theta]}{4\times 2n}\right) - \tfrac{1}{4}\cos(\tau-2\alpha\pi)\sin(3\theta) \\ & + \sum_{n=1}^{\infty}(-1)^{n+1}2\cos[(2n+1)\tau-2\alpha\pi]\left(\frac{\sin[(2n+3)\theta]}{4(2n+2)} - \frac{\sin[(2n-1)\theta]}{4\times 2n}\right) \\ & + \sum_{n=0}^{\infty}(-1)^{n+1}2\sin[(2n+2)\tau-2\alpha\pi] \\ & \times\left(\frac{\sin[(2n+4)\theta]}{4(2n+3)} - \frac{\sin(2n\theta)}{4(2n+1)}\right)\Bigg]. \end{aligned} \tag{A2.6}$$

Through a long and tedious calculation we find

$$\begin{aligned} f_1^{(2)} + f_1^{(7)} &= \tfrac{1}{2}\cos(2\theta-2\alpha\pi) - \tfrac{1}{2}\cos(\tau-\theta) \\ &\quad \times[\tfrac{1}{2}\pi\cos(2\alpha\pi) - \sin(2\alpha\pi)\cosh^{-1}|\sec(\tau+\theta)|] \end{aligned} \tag{A2.7}$$

$$\begin{aligned} f_1^{(4)} + f_1^{(6)} &= \tfrac{1}{4}\sin(\tau+3\theta-2\alpha\pi) - \tfrac{1}{2}\cos(\tau-\theta) \\ &\quad \times[(\tau+\theta)\cos(2\alpha\pi) - \sin(2\alpha\pi)\ln|2\cos(\tau+\theta)|] \end{aligned} \tag{A2.8}$$

$$f_1^{(3)} + f_1^{(5)} = -\tfrac{1}{4}\sin(\tau+3\theta-2\alpha\pi). \tag{A2.9}$$

There are seven terms in the large square bracket of (A2.6), represented by $f_1^{(j)}$, where the superscript j represent the ordinal number of the term. Substituting (A2.7)–(A2.9) into (A2.6) we obtain (60).

References

Aharonov Y, Au C K, Lerner E C and Liang J Q 1984 *Phys. Rev.* D **29** 2396
Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
Brown R A 1985 *J. Phys. A: Math. Gen.* **18** 2497
Gauthier N and Rochon P 1985 *J. Math. Phys.* **26** 2218
Mclachlan N W 1947 *Theory and Applications of Mathieu Functions* (Oxford: Oxford University Press)